

On the scaling ratios for Siegel disks

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Abstract: The boundary of the Siegel disk of a quadratic polynomial with an irrationally indifferent fixed point and the rotation number whose continued fraction expansion is preperiodic has been observed to be self-similar with a certain scaling ratio. The restriction of the dynamics of the quadratic polynomial to the boundary of the Siegel disk is known to be quasisymmetrically conjugate to the rigid rotation with the same rotation number. The geometry of this self-similarity is universal for a large class of holomorphic maps. A renormalization explanation of this universality has been proposed in the literature.

In this paper we provide an estimate on the quasisymmetric constant of the conjugacy, and use it to prove bounds on the scaling ratio λ of the form

$$\alpha^\gamma \leq |\lambda| \leq \delta^s,$$

where s is the period of the continued fraction, and $\alpha \in (0, 1)$ depends on the rotation number in an explicit way, while $\delta, \gamma \in (0, 1)$ have an explicit dependency only on the maximum of the integers in the continued fraction expansion of the rotation number.

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1. Introduction

Definition 1. Given $\theta \in (0, 1]$, the quadratic polynomial P_θ is defined as

$$P_\theta(z) = e^{2i\pi\theta} z \left(1 - \frac{z}{2}\right).$$

Here, the number θ has a unique continued fraction expansion

$$\theta = [a_1, a_2, a_3, \dots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}.$$

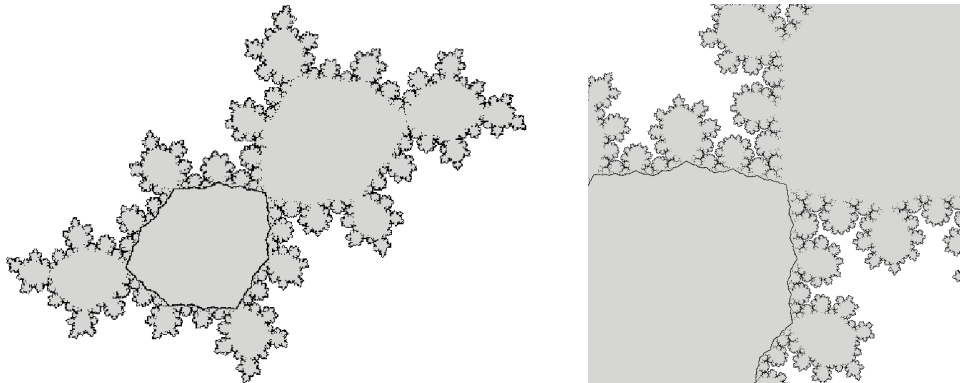


Fig. 1. The Julia set of the golden mean quadratic polynomial, together with a blow up around the critical point. The Siegel disk is bounded by the black curve.

For $\theta \in (0, 1]$, we denote, as usual,

$$\frac{p_n}{q_n} = [a_1, a_2, a_3, \dots, a_n]$$

the n -th best rational approximant of θ , obtained by truncating the continued fraction expansion for θ .

In his classical work [25], Siegel demonstrated that the polynomial P_θ is conformally conjugate to the linear rotation $R_\theta(z) = e^{2i\pi\theta}z$ in a neighborhood of zero if θ is a Diophantine number. The maximal domain of this conjugacy is called the *Siegel disk*. We will denote it as Δ_θ . Bruno demonstrated in [8, 9] that an analytic germ with an irrationally neutral multiplier $f'(0) = e^{2i\pi\theta}$ is conformally linearizable in a neighborhood of zero if θ is a Bruno number, satisfying $\sum_{n=0}^{\infty} q_n^{-1} \ln q_{n+1} < \infty$. Yoccoz proved in [32] that the Bruno condition is also necessary.

Numerical experiments (ex, [21] and [33]) demonstrated that the boundaries of the Siegel disks of analytic germs f with a multiplier $f'(0) = e^{2i\pi\theta^*}$, $\theta^* = \frac{\sqrt{5}-1}{2}$, seemed to be non-differentiable Jordan curves which clearly exhibited a self-similar structure in the neighborhood of the *critical point* $c = 1$. Furthermore, the quantifiers of this self-similarity seemed to be *universal* - independent off the particular choice of f .

The issue of nature of this curve was first addressed by Herman [17] and Świątek [27]. They proved that if θ is of *bounded type*, that is

$$\sup a_i < \infty,$$

then $\partial\Delta_\theta$ is a quasicircle (an image of the circle $|z| = 1$ under a quasi-conformal map) containing the critical point 1. Petersen [23] proved that the Julia set $J(P_\theta)$ is locally connected and has Lebesgue measure zero.

Some numerical observations of [21] and [33] about the self-similarity of $\partial\Delta_\theta$ have been proved analytically by McMullen in [22]. In particular he proved that if θ is preperiodic, then the small-scale dynamics of P_θ admits an asymptotic: certain high order iterates of P_θ , appropriately rescaled, converge. Among other things, using renormalization for commuting holomorphic pairs, McMullen showed that if θ is a quadratic irrational, then the boundary of the Siegel disk is self-similar around the critical point.

Universal nature of the self-similarity can be explained if one could prove hyperbolicity of renormalization in an appropriate functional class. Hyperbolicity of renormalization for golden mean germs has been addressed in [16], where the *cylinder renormalization* operator of [28] is used to demonstrate the existence of an unstable manifold in an appropriate Banach manifold of golden mean germs; furthermore, the authors give an outline of a computer assisted-prove of existence of a stable renormalization manifold, based on rigorous computer-assisted uniformization techniques of [15]. Hyperbolicity of renormalization is accessible analytically for golden-like rotation numbers $\theta = [N, N, N, \dots]$, where N is large; specifically, [29] uses the results of [24] to prove the hyperbolicity conjecture for “close-to-parabolic” rotation numbers which contain a subsequence $a_{i(k)}$ of large integers in the continued fraction expansion.

L. Carleson, based on the earlier numerical observations, has conjectured in [11] that the closest returns to the critical point of the golden mean Siegel disk converge on two lines separated by an angle equal to $\frac{2}{3}\pi$, and suggested an approach to prove this. The numerical results of [21] indicate, however, that close to the

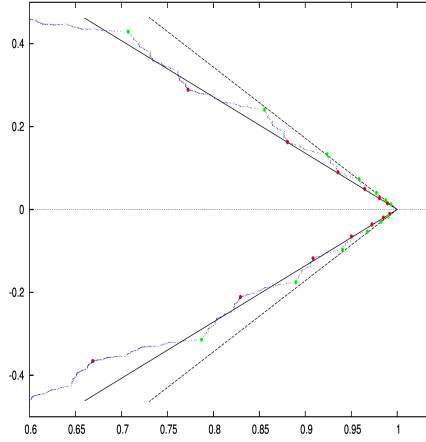


Fig. 2. The geometry of the forward (red) and backward (green) closest returns in the golden mean Siegel disk. The two sets of lines are separated by $107.2\dots^\circ$ and $119.6\dots^\circ$ angles, respectively.

critical point, $\partial\Delta_\theta$ is contained between two sectors of angles $107.2\dots^\circ$ and $119.6\dots^\circ$ (see Fig. 2), the upper bound being definitely less than $\frac{2}{3}\pi$ (which does seem to be a numerical inaccuracy).

Several specific questions of self-similarity of Siegel disks have been addressed in [10]. We will now give a brief summary of the results therein. Recall that a number $\theta \in (0, 1]$ is a quadratic irrational iff its continued fraction is preperiodic: there exists an integer $N \geq 1$ such that $a_{i+s} = a_i$ for all $i \geq N$ and for some integer $s \geq 1$. Given such θ , denote

$$\theta_i = [a_i, a_{i+1}, a_{i+2}, \dots], \quad \alpha = \theta_{N+1}\theta_{N+2}\dots\theta_{N+s}. \quad (1)$$

Theorem 1. (*X. Buff, C. Henriksen*) Let θ be a quadratic irrational. Suppose that $\lambda \in \mathbb{D} \setminus \{0\}$ is the scaling ratio for the self-similarity of the Siegel disk Δ_θ about the critical point. Then

$$\alpha < |\lambda| < 1,$$

where α is as above.

The second result of [10] goes in the direction of Carleson's conjecture.

Theorem 2. (*X. Buff, C. Henriksen*) If $-\pi/\ln(\alpha^2) > 1/2$ then Δ_θ contains a triangle with a vertex at the critical point c . In particular, Δ_{θ^*} , where $\theta^* = (\sqrt{5}-1)/2$ is the golden mean, contains a triangle with one vertex at c .

The hypothesis of Theorem 2 does not hold, for example, for $\theta = [N, N, N, \dots]$ with $N \geq 24$. For such rotation numbers the method of [10] does not provide a lower bound on the angle in the Siegel disk Δ_θ .

X. Buff and C. Henriksen also put forward several questions, one of them being:

Question. Are there bounds on the scaling parameter λ of the form $\delta_1^s \leq |\lambda| \leq \delta_2^s$?

In this paper we answer this question, specifically, we prove the following result.

Main Theorem. Suppose that $\theta = [a_1, a_2, \dots]$ is a quadratic irrational whose continued fraction is eventually periodic with period s , and suppose that $\{p_n/q_n\}$ is the sequence of the best rational approximants of θ . Then there exists $C(n) > 1$, with $\lim_{n \rightarrow \infty} C(n) = 1$, and constants $A > 1$, $\beta > 1$ and $0 < \gamma < 1$, dependent only on $\max\{a_i\}$, such that the following holds.

1) If s is odd, then

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq \frac{C(n)}{\sqrt{A\beta^{-\log_2 \frac{\alpha^2}{1-\alpha^2}} + 1}}, \quad \text{if } \alpha > \frac{1}{\sqrt{2}},$$

and

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq C(n)\beta^{-\frac{1}{2}[s \log_2 \vartheta]}, \quad \text{if } \alpha \leq \frac{1}{\sqrt{2}}.$$

2) If s is even, then

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq \frac{C(n)}{A\beta^{-\log_2 \frac{\alpha}{1-\alpha}} + 1}, \quad \text{if } \alpha > \frac{1}{2},$$

and

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq C(n)\beta^{-[s \log_2 \vartheta]}, \quad \text{if } \alpha \leq \frac{1}{2},$$

where

$$\vartheta = \alpha^{-\frac{1}{s}}.$$

Remark 1. Notice, that

$$\vartheta = \alpha^{-\frac{1}{s}} = (\theta_{N+1}\theta_{N+2}\dots\theta_{N+s})^{-\frac{1}{s}} \geq \max_{1 \leq i \leq s} \{\theta_{N+i}\}^{-1} \geq \frac{1}{[1, B, 1, B, 1, B, \dots]} \equiv \vartheta_B,$$

where $B = \max_{i \geq 1} a_i$. Therefore, the scaling ratios can be bounded from above by s -th powers of constants depending only on B :

$$\frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq C(n) \left(\frac{1}{\beta^{\log_2 \vartheta_B}} \right)^s;$$

thus making the upper bound dependent solely on the period and the maximum of the integers in the continued fraction expansion.

In Section 3 we will give explicit bounds on all constants in the above theorem. First, however, we will give a brief outline of the theory involved in the proofs, and quote several results from the literature that we will require.

2. Preliminaries

2.1. Self-similarity of the Siegel disk. Below, the rotation number θ will be always a quadratic irrational (recall (1)). McMullen demonstrates in [22] that for $n \geq N$

$$\{q_{n+s}\theta\} = (-1)^s \alpha \{q_n\theta\}.$$

In a neighborhood of 1, the map

$$z \mapsto \begin{cases} z^\alpha, & s \text{ is even,} \\ \bar{z}^\alpha, & s \text{ is odd,} \end{cases}$$

conjugates $R_\theta^{q_n}$ to $R_\theta^{q_{n+s}}$ for all $n \geq N$. Furthermore, McMullen introduces in [22] the mapping

$$\psi(z) = \begin{cases} \phi^{-1}(\phi(z)^\alpha), & s \text{ is even,} \\ \phi^{-1}(\overline{\phi(z)}^\alpha), & s \text{ is odd,} \end{cases}$$

where ϕ is the conformal isomorphism of the unit disk with Δ_θ , normalized such that $\phi(1) = 1$, and proves that there exists a neighborhood U of 1, and a constant ϵ such that ψ is well defined on $U \cap \overline{\Delta_\theta}$, conjugates $P_\theta^{q_n}$ to $P_\theta^{q_{n+s}}$, and is $C^{1+\epsilon}$ -conformal or anticonformal:

$$\psi(z) = \begin{cases} 1 + \lambda(z-1) + O(|z-1|^{1+\epsilon}), & s \text{ is even,} \\ 1 + \lambda(\overline{z-1}) + O(|z-1|^{1+\epsilon}), & s \text{ is odd.} \end{cases}$$

The linearization of ψ at 1 will be called Λ :

$$\Lambda(z) = \begin{cases} 1 + \lambda(z-1), & s \text{ is even,} \\ 1 + \lambda(\overline{z-1}), & s \text{ is odd.} \end{cases}$$

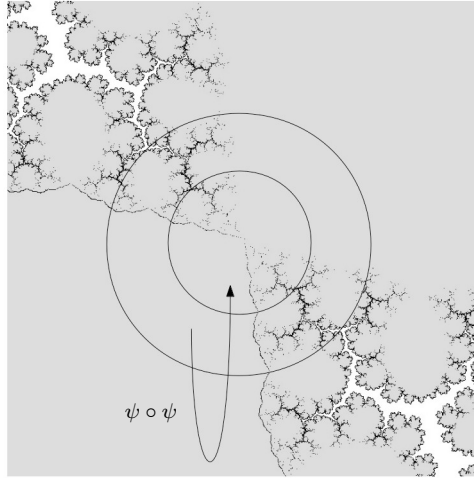


Fig. 3. Self-similarity of the Siegel disk.

Since P_θ is a quadratic polynomial, Δ_θ has one preimage $\Delta_{\theta'}$, different from and symmetric to Δ_θ with respect to 1. McMullen proves in [22] that the blow-ups $\Lambda^{-n}(\Delta_\theta)$ and $\Lambda^{-n}(\Delta_{\theta'})$ converge in the Hausdorff topology on compact subsets of the sphere, to Λ -invariant quasidisks \mathcal{D} and \mathcal{D}' , respectively. The boundaries of both of these quasidisks pass through 1 and ∞ .

Consider the cylinders $C = \mathcal{D}/\Lambda^2$ and $C' = \mathcal{D}'/\Lambda^2$ (we consider Λ^2 instead of Λ , because if s is odd, then Λ is orientation reversing). These two cylinders are conformally equivalent. Buff and Henriksen prove the following Lemma in [10] about the module of these cylinders, which plays an important role in their proof of Theorem 1.

Lemma 1. (*X. Buff, C. Henriksen*) *The module of the cylinder $C = \mathcal{D}/\Lambda^2$ is equal to $-\pi/\ln \alpha^2$.*

We will also use this Lemma in our proof of the lower bound on the scaling ratio in Section 5.

2.2. The Blaschke model of the Siegel disk. We will now present the construction of a Blaschke product model for a quadratic polynomial of Douady, Ghys, Herman and Shishikura. Our description will generally follow that of [30]. The reader is referred to this work for a more detailed description of the Blaschke model for the filled Julia set of a quadratic polynomial.

Define

$$Q^t(z) = e^{2\pi it} z^2 \left(\frac{z-3}{1-3z} \right).$$

The restriction $Q^t|_{\mathbb{T}}$ is a critical circle map with the cubic critical point at $z = 1$, and the critical value $e^{2\pi t}$. By monotonicity, for each irrational $0 < \theta < 1$ there exists $t(\theta)$, such that the rotation number $\rho(Q^{t(\theta)}|_{\mathbb{T}}) = \theta$. Recall, that by the result of Yoccoz [31], any critical circle map with an irrational rotation number is topologically conjugate to the rotation R_θ .

$Q^{t(\theta)}$ has a superattracting fixed points at 0 and ∞ and a double critical point at 1. $Q^{t(\theta)}$ acts as a double branched covering of the immediate basin of attraction of ∞ , $\mathcal{B}(\infty)$. Since $Q^{t(\theta)}$ commutes with the reflection $T(z) = \bar{z}^{-1}$, it also acts as a degree 2 covering on the immediate basin of attraction of the origin, $\mathcal{B}(0)$.

Fix an irrational number $0 < \theta < 1$ of bounded type. By the theorem of Herman and Świątek, the unique homeomorphism $h : \mathbb{T} \rightarrow \mathbb{T}$ with $h(1) = 1$ which conjugates $Q^{t(\theta)}|_{\mathbb{T}}$ to R_θ is quasimetric. Let H be some homeomorphic extension of h to the unit disk. One can, for example, choose the Douady-Earle extension of circle homeomorphisms (cf [14]), or Ahlfors-Beurling extension (cf [19]). In particular, the latter is quasiconformal; its constant of quasiconformality M can be estimated in terms of the quasimetric constant K of h (cf [19]) as

$$M \leq 2K - 1. \tag{2}$$

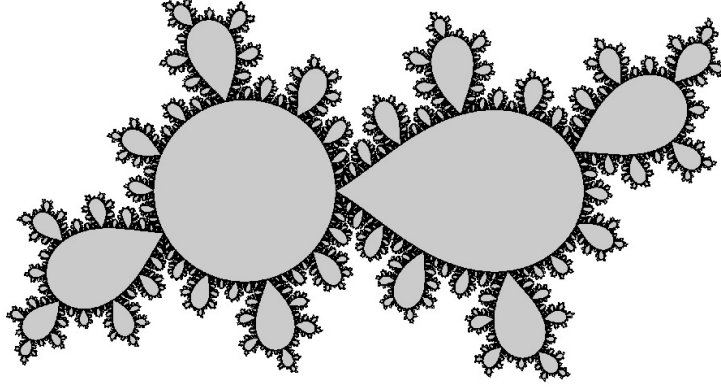


Fig. 4. The filled Julia set of the modified Blaschke product with the golden mean rotation number.

Define the *modified Blaschke product*

$$\bar{Q}_\theta(z) = \begin{cases} Q^{t(\theta)}(z), & |z| \geq 1, \\ H^{-1}(R_\theta(H(z))), & |z| \leq 1, \end{cases}$$

the two definitions matching along the boundary of \mathbb{D} . \bar{Q}_θ is a degree 2 branched covering of the sphere, holomorphic outside of \mathbb{D} , and is quasiconformally conjugate to a rigid rotation on the unit disk. We further conjugate $\bar{Q}_\theta(z)$ by a Möbius transformation m

$$m(z) = \frac{(1 - \bar{a})(z - a)}{(1 - a)(1 - \bar{a}z)}, \quad a = H^{-1}(0), \quad (3)$$

to place $H^{-1}(0)$ at the origin, and set $\tilde{Q}_\theta = m \circ \bar{Q}_\theta(z) \circ m^{-1}$. Define the filled Julia set of \tilde{Q}_θ by

$$K(\tilde{Q}_\theta) = \left\{ z \in \mathbb{C} : \text{the orbit of } \{\tilde{Q}_\theta^{o n}\}_{n \geq 0} \text{ is bounded} \right\},$$

and the Julia set

$$J(\tilde{Q}_\theta) = \partial K(\tilde{Q}_\theta).$$

When the rotation number θ is irrational of bounded type, the action of \tilde{Q}_θ is conjugate to that of a quadratic polynomial. This follows from the following *quasiconformal surgery* due to Douady, Ghys, Herman and Shishikura (cf [13]).

Suppose that $0 < \theta < 1$ is an irrational of bounded type, and H is a quasiconformal extension of the quasisymmetric conjugacy on the circle. Define a new conformal structure on the plane, σ_θ , invariant under \tilde{Q}_θ , as follows. Let σ_0 be the standard conformal structure on \mathbb{C} , and let σ_θ be its pull-back, $H^*\sigma_0$, under H . Since R_θ preserves the standard conformal structure, \tilde{Q}_θ preserves σ_θ on \mathbb{D} . Next, for every $n \geq 1$ pull $\sigma_\theta|_{\mathbb{D}}$ back by $\tilde{Q}_\theta^{o n}$ on the union of all n -th preimages of \mathbb{D} under \tilde{Q}_θ , different from \mathbb{D} . Notice, that since \tilde{Q}_θ is holomorphic outside of the unit disk, the dilatation of the pull-backs of σ_θ will not be increased. Finally, set $\sigma_\theta = \sigma_0$ outside of all preimages of \mathbb{D} . Such σ_θ has a bounded dilatation and is \tilde{Q}_θ invariant. Therefore, by the Measurable Riemann Mapping Theorem (cf [2], [6] and [7]), there exists a unique quasiconformal homeomorphism $f : \mathbb{C} \mapsto \mathbb{C}$, $f(\infty) = \infty$, $f(0) = 0$ and $f(1) = 1$, such that $f^*\sigma_0 = \sigma_\theta$. Set

$$f_\theta = f \circ \tilde{Q}_\theta \circ f^{-1}.$$

This f_θ is a self-map of the sphere that preserves σ_0 , therefore it is holomorphic. It is, furthermore, a proper map of degree 2 (since \tilde{Q}_θ is), therefore it is a quadratic polynomial. Since $f_\theta|_{f(\mathbb{D})}$ is quasiconformally conjugate to a rigid rotation, $f(\mathbb{D})$ contains a Siegel disk for f_θ . Since $f(1) = 1$ is a critical point of f_θ , $\overline{\{f_\theta^{on}(1)\}_{n \geq 0}}$ is the boundary of the Siegel disk, while $\{f_\theta^{on}(1)\}_{n \geq 0}$ itself is also dense in $f(\mathbb{T})$, therefore $f(\mathbb{T})$ is the boundary of the Siegel disk. Due to our normalization of f , we must have

$$f_\theta = P_\theta. \quad (4)$$

The above discussion is a sketch of the proof of the following theorem.

Theorem 1. (*Douady, Ghys, Herman, Shishikura*) *Let f_θ be a quadratic polynomial which has a fixed Siegel disk Δ of rotation number θ of bounded type. Then f_θ is quasiconformally conjugate to the modified Blaschke product \tilde{Q}_θ . In particular, $\partial\Delta$ is a quasicircle passing through the critical point of f_θ .*

2.3. Bounds on the quasiconformal distortion of angles. For $0 < r < 1$, let $\mu(r)$ be the module of the unit disk slit along the real axis from 0 to r - the so called *Grötzsch's extremal domain*. The module function has the following explicit expression (cf [18])

$$\mu(r) = \frac{1}{4} \frac{K'(r)}{K(r)}, \quad (5)$$

where $K(r)$ is the elliptic integral of the first kind

$$K(r) = \int_0^r \frac{dx}{\sqrt{1-x^2}\sqrt{1-r^2x^2}},$$

and $K'(r) = K(\sqrt{1-r^2})$. In particular, the following asymptotic holds as $r \rightarrow 0$ (cf [18])

$$\mu(r) = \frac{1}{2\pi} \ln \frac{4}{r} + O(r^2), \quad \mu(r) \leq \frac{1}{2\pi} \ln \frac{4}{r}. \quad (6)$$

Next, given $M \geq 1$, set

$$\phi_M(r) = \mu^{-1}(M\mu(r)). \quad (7)$$

The “distortion function” ϕ_M is continuous and strictly increasing in $(0, 1)$, with $\phi_M(0) = 0$ and $\phi_M(1) = 1$ (cf [1]). Furthermore, $\phi_M(t) \leq t$.

The following result from [1] will be important for our estimates

Quasiconformal Distortion of Angles Theorem. (*S. B. Agard, F. W. Gehring*) *Suppose that f is a M -quasiconformal mapping of the extended plane, and that $f(\infty) = \infty$. Then for each triple of distinct finite points z_1, z_0, z_2 ,*

$$\sin \frac{\beta}{2} \geq \phi_M \left(\sin \frac{\alpha}{2} \right),$$

where ϕ_M is as in (2.3), and

$$\alpha = \arcsin \left(\frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|} \right), \quad \beta = \arcsin \left(\frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_0)| + |f(z_2) - f(z_0)|} \right). \quad (8)$$

The inequality is sharp.

2.4. *The quasimetric property and moduli.* We start with the definition of the quasimetric property.

Definition 2. Let $\eta : [0, \infty) \rightarrow [0, \infty)$ be an increasing homeomorphism, $A \subset \mathbb{C}$, and $h : A \rightarrow \mathbb{C}$ a mapping. f is η -quasimetric if for each triple $z_0, z_1, z_2 \in A$

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta \left(\frac{|z_0 - z_1|}{|z_0 - z_2|} \right).$$

A homeomorphism $h : \mathbb{R} \rightarrow \mathbb{R}$ is quasimetric iff for every real x and $\delta > 0$

$$\left| \frac{h(x + \delta) - h(x)}{h(x) - h(x - \delta)} \right| \leq K.$$

Define the cross-ratio of any four points $a \leq b \leq c \leq d$ on the real line as

$$\mathbf{Cr}(a, b, c, d) = \frac{|a - b| \cdot |c - d|}{|a - c| \cdot |d - b|}.$$

An important property of the cross-ratio is that it is preserved under linear-fractional transformations.

Following [26], we introduce a more general class of functions which play a role similar to the cross-ratio.

Definition 3. A cross-ratio module is a function χ from all quadruples of points on the real line which satisfy $a \leq b \leq c \leq d$, or $a \geq b \geq c \geq d$, with values in $[0, \infty)$, provided that

- there is a constant C such that if $\mathbf{Cr}(a, b, c, d) \geq 1/4$, then $\chi(a, b, c, d) \geq C$;
- for every $\epsilon > 0$ there is a $\delta > 0$ so that if $\mathbf{Cr}(a, b, c, d) < \delta$, then $\chi(a, b, c, d) < \epsilon$.

Example 1. Consider a Jordan curve $\gamma \in \hat{\mathbb{C}}$ with four marked points on it, A, B, C and D . The four points divide the curve into four arcs. Such configuration defines a quadrilateral $Q(A, B, C, D)$. Choose one of the arcs to be the *base*, and map one of the components of the complement of γ conformally onto the rectangle with the vertices $(0, 1, 1 + ia, ia)$, so that the base is mapped onto $(0, 1)$. The parameter a is called the module of the quadrilateral, and denoted $\text{mod } Q(A, B, C, D)$.

In the particular case when $\gamma = \mathbb{R}$, set

$$\chi(A, B, C, D) = \frac{1}{\text{mod } Q(A, B, C, D)}. \quad (9)$$

If $A = B$ or $C = D$, set $\chi(A, B, C, D) = 0$.

The module of a quadrilateral is related to the module of a *ring domain*. Let A, B, C and D be points on the real line, and let v be a Jordan arc connecting A and D in the upper half plane. Intervals (A, B) , (B, C) , (C, D) and the arc v define a quadrilateral Q' with the base (A, B) . Let Q'' be its reflection with respect to the real line. The union of the interiors of Q' , Q'' and the intervals (A, B) and (C, D) define a ring domain of $R(Q')$ associated with Q' . The moduli of a quadrilateral and of the associated ring domain are related as

$$\text{mod } R(Q') = \frac{1}{2 \text{mod } Q'}.$$

This can be seen from the following argument. Map the ring domain $R(Q')$ conformally to an annulus with the radii 1 and r , whose module is $(2\pi)^{-1} \ln r$. The same conformal transformation maps the quadrilateral Q' to the upper semi-annulus. A further application of the logarithm, maps the semi-annulus to a quadrilateral $(0, \ln r, \ln r + i\pi, i\pi)$ (notice, that the base (A, B) is mapped to $(0, \ln r)$). The module of the last rectangle is $\pi / \ln r$.

We will now describe how the moduli of the quadrilaterals Q and Q' are related. Consider a family of *admissible curves* - any collection of curves that are contained inside the quadrilateral and join the base with the opposite side. If the any set of admissible curves for a quadrilateral Q_1 is contained in some set of admissible curves of Q_2 , then $\text{mod } Q_1 \geq \text{mod } Q_2$. On the other hand, one can also bound the module of Q_1 in terms of that of Q_2 :

Lemma 2. (Lemma 2.1 in [26]) Suppose that points A, B, C, D are cyclically ordered on \mathbb{R} . For every $k > 0$, if $|D - A| \leq k/2$, then there is a Jordan arc γ with the endpoints D and A , such that the quadrilateral Q' with sides (A, B) , (B, C) , (C, D) and γ has the interior disjoint from $\{z \in \mathbb{C} : \Im z \geq k\}$, and satisfies

$$\text{mod } Q' \leq \frac{1}{1 - \frac{|D-A|}{k}} \text{ mod } Q.$$

We will now show that χ is indeed a cross-ratio module. If $\mathbf{Cr}(A, B, C, D) \geq 1/4$ then

$$\left(1 + \frac{|B - C|}{|A - B|}\right) \left(1 + \frac{|B - C|}{|C - D|}\right) \leq 4 \implies |A - B| \geq \frac{1}{3}|B - C| \quad \text{and} \quad |C - D| \geq \frac{1}{3}|B - C|,$$

and, therefore, the ring domain $R(A, B, C, D)$ contains a ring with module at least

$$\text{mod } R \geq \frac{1}{2\pi} \ln \frac{1/2 + \frac{1}{3}}{1/2} = \frac{1}{2\pi} \ln \frac{5}{3},$$

therefore, if Q' is the quadrilateral with the sides (A, B) , (B, C) , (C, D) and a semicircular arc γ connecting points A and D , we get that

$$\chi(A, B, C, D) = \frac{1}{\text{mod } Q(A, B, C, D)} \geq \frac{1}{\text{mod } Q'} = 2 \text{ mod } R(Q') \geq \frac{1}{\pi} \ln \frac{5}{3}. \quad (10)$$

The second condition in the definition of the cross-ratio module follows from the above Lemma and Teichmüller's Module Theorem (cf page 56 in [20]).

Teichmüller's Module Theorem. If a ring domain R separates points 0 and z_1 from z_2 and ∞ , then

$$\text{mod } R \leq 2\mu \left(\sqrt{\frac{|z_1|}{|z_1| + |z_2|}} \right),$$

where μ is as in (5).

Now, we can demonstrate the second property of the cross-ratio module. Assume that $\mathbf{Cr}(A, B, C, D) < \delta$. This inequality implies that

$$\min \left\{ \frac{|A - B|}{|C - B|}, \frac{|C - D|}{|C - B|} \right\} < \delta^{\frac{1}{2}} / (1 - \delta^{\frac{1}{2}}) = \delta'. \quad (11)$$

By Lemma 2, there exists a quadrilateral Q' with the base (A, B) , sides (B, C) , (C, D) , and an arc γ connecting A and D , disjoint from $\{z \in \mathbb{C} : \Im z \geq 2|D - A|\}$, whose module satisfies

$$2 \text{ mod } Q \geq \text{mod } Q'.$$

By Theorem 1,

$$\text{mod } R(Q') < 2\mu \left(\sqrt{\frac{1}{1 + \delta'}} \right),$$

and

$$\chi(A, B, C, D) = \frac{1}{\text{mod } Q} \leq \frac{2}{\text{mod } Q'} \leq 4 \text{ mod } R(Q') \leq 8\mu \left(\sqrt{\frac{1}{1 + \delta'}} \right) = 8\mu \left(\sqrt{1 - \delta^{\frac{1}{2}}} \right) \equiv \epsilon. \quad (12)$$

A configuration of n quadruples of points (a_i, b_i, c_i, d_i) , $i = 1, \dots, n$, will be called *allowable*, if the intervals a_i, d_i are pairwise disjoint modulo 1 and $d_i - a_i < 1$. We will also say that a configuration of n quadruples of points (a_i, b_i, c_i, d_i) , $i = 1, \dots, n$, has the *intersection number* k if the supremum over points $x \in \mathbb{R}$ of k' such that x is contained modulo 1 in k' intervals from the configuration, is equal to k .

Definition 4. Let $g : \mathbb{R} \mapsto \mathbb{R}$ be strictly increasing and suppose that $g(x) - x$ is 1-periodic. Let χ be a cross-ratio module. We say that f satisfies the cross-ratio inequality with respect to χ with bound Q iff for any choice of quadruples of points (a_i, b_i, c_i, d_i) , $i = 1, \dots, n$, in an allowable configuration the following holds:

$$\prod_{i=1}^n \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq Q.$$

We mention the following Lemma without the proof.

Lemma 3. (Lemma 1.1. from [26]) Suppose that g satisfies the cross-ratio inequality with respect to χ with the bound Q . Let (a_i, b_i, c_i, d_i) be any configuration of quadruples of points with the intersection number k . Then

$$\prod_{i=1}^n \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq Q^{2k}.$$

3. Statement of results

We are now ready to state the main result of the paper in a more detailed form.

Main Theorem. Suppose that $\theta = [a_1, a_2, \dots]$, $a_i < B$, is a quadratic irrational whose continued fraction is eventually periodic with period s , and suppose that $\{p_n/q_n\}$ is the sequence of the best rational approximants of θ . Then there exists $C(n) > 1$, with

$$\lim_{n \rightarrow \infty} C(n) = 1,$$

such that the following holds.

1) If s is odd, then

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq \frac{C(n)}{\sqrt{K \left(\frac{1}{1+K^{-1}} \right)^{\log_2 \frac{\alpha^2}{1-\alpha^2}} + 1}}, \quad \text{if } \alpha > \frac{1}{\sqrt{2}}, \quad (13)$$

and

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq C(n) (1 + K^{-1})^{-\frac{1}{2}[s \log_2 \vartheta]}, \quad \text{if } \alpha \leq \frac{1}{\sqrt{2}}. \quad (14)$$

2) If s is even, then

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq \frac{C(n)}{K \left(\frac{1}{1+K^{-1}} \right)^{\log_2 \frac{\alpha}{1-\alpha}} + 1}, \quad \text{if } \alpha > \frac{1}{2}, \quad (15)$$

and

$$\alpha^\gamma \leq \frac{|P_\theta^{q_{n+s}}(1) - 1|}{|P_\theta^{q_n}(1) - 1|} \leq C(n) (1 + K^{-1})^{-[s \log_2 \vartheta]}, \quad \text{if } \alpha \leq \frac{1}{2}. \quad (16)$$

In the above bounds,

$$K = \lambda(2K_2 - 1)^{4K_2-2} K_2^{2K_2-1}, \quad K_2 = \max \{2, K_1^{B+1}\}, \quad K_1 = 2 \left(1 - \left(\mu^{-1} \left(\frac{1}{87\pi} \ln \frac{5}{3} \right) \right)^2 \right)^{-6},$$

where μ is the module (5) of the Grötzsch's extremal domain, $\lambda(M) \leq e^{\pi M}/16$ is the "circular distortion", defined as

$$\lambda(M) = \sup_{0 \leq \phi \leq 2\pi} \{|f(e^{i\phi})| : f \in \mathcal{F}\}, \quad \mathcal{F} = \{f : \mathbb{C} \mapsto \mathbb{C}, M\text{-quasiconformal}, f(0) = 0, f(1) = 1\},$$

while

$$\nu = \left(\frac{1+\alpha}{\alpha} \right)^{\frac{1}{s}}, \quad \vartheta = \alpha^{-\frac{1}{s}},$$

and

$$\gamma = 1 - \frac{16}{\pi} \left(\frac{2 - \sqrt{2 + \sqrt{2}}}{64} \right)^{\frac{2K_2-1}{2}} \left(1 + O \left(\left(\frac{2 - \sqrt{2 + \sqrt{2}}}{64} \right)^{2K_2-1} \right) \right).$$

4. An upper bound on the scaling ratio

The following is a version of the Hölder continuity property for quasimetric homeomorphisms. The Hölder property is a classical result (cf [3], [12]); here we will prove a Lemma adopted for our situation.

Lemma 4. *Let $h : T \mapsto h(T)$ be a quasimetric homeomorphism with a quasimetric constant K .*

1) *If $I \subseteq J$ are two intervals in T sharing a single boundary point c , and satisfying $|I|/|J| = \alpha \leq 1/2$, then*

$$\left(\frac{1}{1+K}\right)^{[s \log_2 \vartheta]+1} \leq \frac{|h(I)|}{|h(J)|} \leq \left(\frac{1}{1+K^{-1}}\right)^{[s \log_2 \vartheta]}, \quad \vartheta = \alpha^{-\frac{1}{s}}.$$

2) *If I and J are two closed intervals in T such that their intersection is a single boundary point $I \cap J = \{c\}$, and satisfying $|I|/|J| = \alpha < 1$, then*

$$\frac{1}{(1+K)^{[s \log_2 \nu]+1} - 1} \leq \frac{|h(I)|}{|h(J)|} \leq \frac{1}{(1+K^{-1})^{[s \log_2 \nu] - 1}}, \quad \nu = \left(\frac{1+\alpha}{\alpha}\right)^{\frac{1}{s}}.$$

3) *If $I \subseteq J$ are two intervals in T sharing a single boundary point c , and satisfying $|I|/|J| = \alpha > 1/2$, then*

$$\frac{1}{K \left(\frac{1}{1+K^{-1}}\right)^{\log_2 \frac{\alpha}{1-\alpha}} + 1} \leq \frac{|h(I)|}{|h(J)|} \leq \frac{1}{K \left(\frac{1}{1+K^{-1}}\right)^{\log_2 \frac{\alpha}{1-\alpha}} + 1}.$$

Proof. 1) *Case $I \subset J$.* Let J_n be the unique closed subinterval of J of length $2^{-n}|J|$ that shares the same end point with I and J . Since f is quasimetric on J with constant K , we have

$$\frac{|h(J_{n+1})|}{|h(J_n)|} \leq \frac{|h(J_{n+1})|}{|h(J_{n+1})| + |h(J_n \setminus J_{n+1})|} \leq \frac{1}{1 + \frac{|h(J_n \setminus J_{n+1})|}{|h(J_{n+1})|}} \leq \frac{1}{1 + K^{-1}}, \quad (17)$$

in a similar way,

$$\frac{1}{1+K} \leq \frac{|h(J_{n+1})|}{|h(J_n)|}.$$

We have $|h(J_n)|/|h(J)| = \prod_{i=0}^{n-1} |h(J_{i+1})|/|h(J_i)|$, therefore,

$$\left(\frac{1}{1+K}\right)^n \leq \frac{|h(J_n)|}{|h(J)|} \leq \left(\frac{1}{1+K^{-1}}\right)^n.$$

Now, set $m = [\log_2 \alpha^{-1}] + 1$, then $J_m \subseteq I \subseteq J_{m-1}$, therefore,

$$\left(\frac{1}{1+K}\right)^{[-\log_2 \alpha]+1} \leq \frac{|h(J_m)|}{|h(J)|} \leq \frac{|h(I)|}{|h(J)|} \leq \frac{|h(J_{m-1})|}{|h(J)|} \leq \left(\frac{1}{1+K^{-1}}\right)^{[-\log_2 \alpha]}.$$

2) *Case $I \cap J = \{c\}$.*

$$\frac{|h(I)|}{|h(J)|} = \frac{|h(I)|}{|h(J \cup I)| - |h(I)|} = \frac{1}{\frac{|h(J \cup I)|}{|h(I)|} - 1},$$

and

$$\frac{1}{(1+K)^{[-\log_2 \frac{\alpha}{1+\alpha}]+1} - 1} \leq \frac{|h(I)|}{|h(J)|} \leq \frac{1}{(1+K^{-1})^{[-\log_2 \frac{\alpha}{1+\alpha}]} - 1}$$

3) *Case $I \subset J$, $|I|/|J| = \alpha > 1/2$.* Let \tilde{I} be a closed subinterval of I of length $|J \setminus I|$ such that the intersection of \tilde{I} with $J \setminus I$ is a single boundary point of both intervals. Let $\hat{I} \supset J \setminus I$ be a closed interval of length $|I|$ sharing one endpoint with I and $J \setminus I$.

$$\frac{|h(I)|}{|h(J)|} = \frac{1}{\frac{|h(J \setminus I)|}{|h(I)|} + 1} = \frac{1}{\frac{|h(J \setminus I)|}{|h(\tilde{I})|} \frac{|h(\tilde{I})|}{|h(I)|} + 1} \geq \frac{1}{K \left(\frac{1}{1+K^{-1}}\right)^{\log_2 \frac{\alpha}{1-\alpha}} + 1},$$

$$\frac{|h(I)|}{|h(J)|} = \frac{1}{\frac{|h(\hat{I})|}{|h(I)|} \frac{|h(J \setminus I)|}{|h(\hat{I})|} + 1} \leq \frac{1}{K^{-1} \left(\frac{1}{1+K}\right)^{\log_2 \frac{\alpha}{1-\alpha} + 1} + 1}.$$

□

With regard to critical circle maps with a rotation number whose continued fraction is eventually periodic, a particular case of which is the dynamical system $P_\theta|_{\partial\Delta_\theta}$, the above Lemma essentially provides bounds on the scaling ratio, of the form $C_1\delta_1^s \leq |\lambda| \leq C_2\delta_2^s$, if one takes J and I to be the intervals $[c, f^{q_n}(c)]$ and $[c, f^{q_{n+s}}(c)]$. The only remaining ingredient is a bound on the constant of quasismymetry. This will be the subject of Section 6, where we will also show that this bound can be taken essentially independent from the rotation number.

We notice, however, that for sufficiently large K , the lower bound on the quasismymetric distortion of a ration of two intervals is of the order $(1+K)^{[-\log_2 \alpha]+1}$ in Cases 1) and 2) in the above Lemma. But

$$\left(\frac{1}{1+K}\right)^{[\log_2 \alpha^{-1}]+1} \leq (1+K)^{\log_2 \alpha} = \alpha^{\frac{1}{\log_2(1+K)}},$$

the power of α in the last expression being larger than 1. This means, that the lower bounds in the Lemma above are worse than the bound $|\lambda| \geq \alpha$ of Buff and Henriksen. In the next Section we will derive a better lower bound on the absolute value of the scaling ratio of the form α^γ with an explicit $0 < \gamma < 1$.

5. A lower bound on the scaling ratio

Recall, that by the result of McMullen [22] the rescalings of the Siegel disk and its preimage converge to Λ -invariant quasidisks \mathcal{D} and \mathcal{D}' .

Consider the cylinder $C = \mathcal{D}/\Lambda^2$. Let f be the quasiconformal conjugacy between the dynamics of the modified Blaschke product \tilde{Q}_θ and that of P_θ on the sphere, as described in Section 2.2.

Let $\gamma_{\min} < \gamma_{\max}$ be any two angles such that the angle γ between any two vectors $f(z_1) - 1$ and $f(z_2) - 1$, $z_i \neq 1$, of equal length, $|f(z_1) - 1| = |f(z_2) - 1|$, and with the end points $f(z_i) \in \Delta_\theta$ on “opposite sides” of the critical point, i.e. $\text{signum}\{\Im(z_1)\} \neq \text{signum}\{\Im(z_2)\}$, is asymptotically bounded by γ_{\min} and γ_{\max} . The boundary of the Siegel disk is, therefore, asymptotically contained between the two sectors \mathcal{S}_{\min} and \mathcal{S}_{\max} with the vertex at 1 and angles γ_{\min} and γ_{\max} , respectively. Set

$$C_{\max} = \mathcal{S}_{\gamma_{\max}}/\Lambda^2, \quad C_{\min} = \mathcal{S}_{\gamma_{\min}}/\Lambda^2.$$

Since the circumference l of the cylinders satisfies

$$|l|^2 = \frac{\text{area } C_{\min}}{\text{mod } C_{\min}} = \frac{\text{area } C_{\max}}{\text{mod } C_{\max}},$$

we get, using Lemma 1

$$\left(\ln \frac{1}{|\lambda|^2}\right)^2 = \frac{\gamma_{\min} \ln \frac{1}{|\lambda|^2}}{\text{mod } C_{\min}} \geq \frac{\gamma_{\min} \ln \frac{1}{|\lambda|^2}}{\text{mod } C} \implies |\lambda| \leq \alpha^{\frac{\gamma_{\min}}{\pi}}, \quad (18)$$

$$\left(\ln \frac{1}{|\lambda|^2}\right)^2 = \frac{\gamma_{\max} \ln \frac{1}{|\lambda|^2}}{\text{mod } C_{\min}} \leq \frac{\ln \frac{1}{|\lambda|^2}}{\text{mod } C} \implies |\lambda| \geq \alpha^{\frac{\gamma_{\max}}{\pi}}. \quad (19)$$

We will now proceed with bounds on γ_{\max} and γ_{\min} .

5.1. Estimate on γ_{\min} . Take, $z_0 = 1$, and any two $z_1, z_2 \in \mathbb{T}$ with $\Im(z_1) > 0$ and $\Im(z_2) < 0$, sufficiently close to 1. Elementary geometric considerations demonstrate that

$$\frac{|z_1 - z_2|}{|z_1 - 1| + |z_2 - 1|} = 1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right),$$

while

$$\begin{aligned} \arcsin\left(\frac{|z_1 - z_2|}{|z_1 - 1| + |z_2 - 1|}\right) &= \frac{\pi}{2} + O(|z_1 - 1| + |z_2 - 1|), \\ \sin\left(\frac{1}{2} \arcsin\left(\frac{|z_1 - z_2|}{|z_1 - 1| + |z_2 - 1|}\right)\right) &= \frac{1}{\sqrt{2}} + O(|z_1 - 1| + |z_2 - 1|). \end{aligned}$$

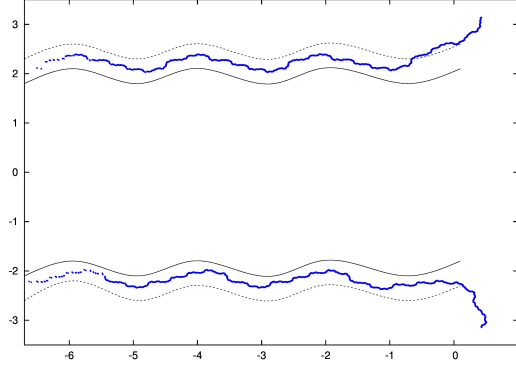


Fig. 5. The postcritical set of the quadratic polynomial with the rotation number $[5, 5, 5, \dots]$ in the neighborhood of the critical point in the logarithmic coordinate $\ln(z-1)$. The distance between the two solid lines is γ_{\min} , that between the dashed is γ_{\max} . The postcritical set is asymptotically contained between the lines.

We now notice, that

$$K' \left(\frac{1}{\sqrt{2}} \right) = K \left(\sqrt{1 - \left(\frac{1}{\sqrt{2}} \right)^2} \right) = K \left(\frac{1}{\sqrt{2}} \right),$$

therefore, it follows from (5), that

$$\mu \left(\frac{1}{\sqrt{2}} + O(|z_1 - 1| + |z_2 - 1|) \right) = \frac{1}{4} + O(|z_1 - 1| + |z_2 - 1|).$$

At the same time, according to (6),

$$\mu^{-1}(x) = 4e^{-2\pi x} + O(e^{-4\pi x}).$$

We, therefore, get that for large M ,

$$\mu^{-1} \left(M\mu \left(\frac{1}{\sqrt{2}} + O(|z_1 - 1| + |z_2 - 1|) \right) \right) = \mu^{-1} \left(\frac{M}{4} + O(|z_1 - 1| + |z_2 - 1|) \right) = 4e^{-M\frac{\pi}{2}} e^{O(|z_1 - 1| + |z_2 - 1|)}.$$

We finally get that

$$\frac{|f(z_1) - f(z_2)|}{|f(z_1) - 1| + |f(z_2) - 1|} \geq \sin \left(2 \arcsin \left(4e^{-\frac{M\pi}{2}} e^{O(|z_2 - z_1|)} \right) \right) = \left(8e^{-\frac{M\pi}{2}} + O \left(e^{-3\frac{M\pi}{2}} \right) \right) e^{O(|z_2 - z_1|)} \quad (20)$$

(we have used that $O(|z_1 - 1| + |z_2 - 1|) = O(|z_2 - z_1|)$).

Since for small $|f(z_1) - f(z_2)|$ (large M), the argument of the arccos below is close to one, and arccos is a decreasing function of its argument, we get

$$\begin{aligned} \arccos \left(\frac{|f(z_1) - 1|^2 + |f(z_2) - 1|^2 - |f(z_1) - f(z_2)|^2}{2|f(z_1) - 1||f(z_2) - 1|} \right) &= \\ \arccos \left(\frac{1}{2} \left(\frac{|f(z_1) - 1|}{|f(z_2) - 1|} + \frac{|f(z_2) - 1|}{|f(z_1) - 1|} - \frac{|f(z_1) - f(z_2)|^2}{|f(z_1) - 1||f(z_2) - 1|} \right) \right) &\geq \\ \arccos \left(\frac{1}{2} \left(\frac{|f(z_1) - 1|}{|f(z_2) - 1|} + \frac{|f(z_2) - 1|}{|f(z_1) - 1|} \right) - \left(\frac{|f(z_1) - 1|}{|f(z_2) - 1|} + \frac{|f(z_2) - 1|}{|f(z_1) - 1|} \right)^2 (32e^{-M\pi} + O(e^{-2M\pi})) e^{O(|z_2 - z_1|^2)} \right) &\geq \\ \arccos \left(\frac{1}{2} (1 + 1) - (1 + 1)^2 (32e^{-M\pi} + O(e^{-2M\pi})) e^{O(|z_2 - z_1|^2)} \right) &\geq \\ \arccos \left(1 - (128e^{-M\pi} + O(e^{-2M\pi})) e^{O(|z_2 - z_1|^2)} \right), \end{aligned}$$

where we have used (20) in the third line. We, therefore, get that

$$\begin{aligned}\gamma_{\min} &= \arccos\left(1 - (128e^{-M\pi} + O(e^{-2M\pi}))\right) \\ &= \sqrt{2}\sqrt{128e^{-M\pi} + O(e^{-2M\pi})} (1 + O(e^{-M\pi})) \\ &= 16e^{-M\frac{\pi}{2}} (1 + O(e^{-M\pi})).\end{aligned}\tag{21}$$

At this point we would like to compare the upper bound $\alpha^{\frac{\gamma_{\min}}{\pi}}$ against the upper bound coming from Lemma 4 and included in the Main Theorem. Notice, that

$$\alpha^{\frac{\gamma_{\min}}{\pi}} = O\left(\left(\vartheta^{-\text{const}} e^{-\pi K}\right)^s\right),\tag{22}$$

while the upper bound from the Main Theorem is of the order

$$O\left(\left(\frac{1}{(1+K^{-1})^{\text{const}}}\right)^s\right),\tag{23}$$

and $1/s$ power of (22) tends to 1 from below super-exponentially fast - much faster than (23). This is the reason for reporting the upper bounds from Lemma 4 in the Main Theorem, rather than the ones supplied by (18).

5.2. Estimate on γ_{\max} . To obtain an upper bound on the angle γ , and hence a lower bound on λ , we take $z_1 \in \mathbb{T}$ and z_2 is in the preimage of \mathbb{T} under the modified Blaschke product, distinct from \mathbb{T} , with $\Im(z_1) > 0$ and $\Im(z_2) > 0$, both sufficiently close to $z_0 = 1$. Elementary geometric considerations demonstrate

$$\begin{aligned}\frac{|z_1 - z_2|}{|z_1 - 1| + |z_2 - 1|} &= \frac{\sqrt{|z_1 - 1|^2 + |z_2 - 1|^2 - 2|z_1 - 1||z_2 - 1|\cos\frac{\pi}{3}}}{|z_1 - 1| + |z_2 - 1|} + O\left((|z_1 - 1| + |z_2 - 1|)^2\right) \\ &= \frac{\sqrt{|z_1 - 1|^2 + |z_2 - 1|^2 - \sqrt{2}|z_1 - 1||z_2 - 1|}}{|z_1 - 1| + |z_2 - 1|} + O\left((|z_1 - 1| + |z_2 - 1|)^2\right).\end{aligned}\tag{24}$$

Next,

$$\begin{aligned}\arccos\left(\frac{|f(z_1) - 1|^2 + |f(z_2) - 1|^2 - |f(z_1) - f(z_2)|^2}{2|f(z_1) - 1||f(z_2) - 1|}\right) &\geq \\ \arccos\left(\frac{|f(z_1) - 1|^2 + |f(z_2) - 1|^2 - (|f(z_1) - 1| + |f(z_2) - 1|)^2 \sin^2(\beta)}{2|f(z_1) - 1||f(z_2) - 1|}\right).\end{aligned}$$

We will now obtain a bound on the angle β (cf Theorem 2.3). To that end we will estimate the expression (24) from below. It is clear that if $|f(z_1) - 1| = |f(z_2) - 1|$, then the ratio $|z_1 - 1|/|z_2 - 1|$ is at most $1/\alpha^2$ and least α^2 , and for $i = 1, 2$ there exist s_i and an integer n , such that

$$|z_i - 1| = s_i \alpha^{2(n+1)} + (1 - s_i) \alpha^{2n},$$

and

$$\begin{aligned}\frac{\sqrt{|z_1 - 1|^2 + |z_2 - 1|^2 - \sqrt{2}|z_1 - 1||z_2 - 1|}}{|z_1 - 1| + |z_2 - 1|} &+ O\left((|z_1 - 1| + |z_2 - 1|)^2\right) = \\ \frac{\sqrt{z^{-2} + z^2 - \sqrt{2}}}{z^{-1} + z} &+ O\left((|z_1 - 1| + |z_2 - 1|)^2\right),\end{aligned}\tag{25}$$

where $z = \sqrt{\frac{s_1 \alpha^2 + 1 - s_1}{s_2 \alpha^2 + 1 - s_2}}$.

The minimum of (25) is achieved at $z = 1$, and is equal to $\sqrt{2 - \sqrt{2}}/2 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)$; its inverse sine being equal to

$$\alpha = \frac{\pi}{8} + O\left((|z_1 - 1| + |z_2 - 1|)^2\right).$$

Next,

$$\sin \frac{\alpha}{2} = \sin \frac{\pi}{16} + O\left((|z_1 - 1| + |z_2 - 1|)^2\right) = \sqrt{\frac{2 - \sqrt{2 + \sqrt{2}}}{4}} + O\left((|z_1 - 1| + |z_2 - 1|)^2\right).$$

At the next step we will evaluate the function ϕ_M at the above value. For brevity of notation, denote

$$\zeta = \sqrt{\frac{2 - \sqrt{2 + \sqrt{2}}}{4}}.$$

According to (6),

$$\begin{aligned} \mu\left(\zeta + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right) &\leq -\frac{1}{2\pi} \ln\left(\frac{\zeta}{4}\right) + O\left((|z_1 - 1| + |z_2 - 1|)^2\right), \\ \phi_M\left(\zeta + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right) &\geq \left(4\left(\frac{\zeta}{4}\right)^M + O\left(\left(\frac{\zeta}{4}\right)^{2M}\right)\right) \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right), \end{aligned}$$

that is,

$$\begin{aligned} \beta &\geq 2 \arcsin\left(\left(4\left(\frac{\zeta}{4}\right)^M + O\left(\left(\frac{\zeta}{4}\right)^{2M}\right)\right) \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right)\right) \implies \\ |f(z_1) - f(z_2)| &\geq (|f(z_1) - f(z_0)| + |f(z_2) - f(z_0)|) \sin\left(2 \arcsin\left(\left(4\left(\frac{\zeta}{4}\right)^M + O\left(\left(\frac{\zeta}{4}\right)^{2M}\right)\right) \right.\right. \\ &\quad \left.\left. \times \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right)\right)\right) \\ &\geq (|f(z_1) - 1| + |f(z_2) - 1|) \left(8\left(\frac{\zeta}{4}\right)^M + O\left(\left(\frac{\zeta}{4}\right)^{2M}\right)\right) \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right). \end{aligned}$$

We now recall that $|f(z_1) - 1| = |f(z_2) - 1|$, and obtain

$$\begin{aligned} \arccos\left(\frac{|f(z_1) - 1|^2 + |f(z_2) - 1|^2 - |f(z_1) - f(z_2)|^2}{2|f(z_1) - 1||f(z_2) - 1|}\right) &\geq \\ \arccos\left(\frac{1}{2} \left(1 + 1 - (1 + 1)^2 \left(8\left(\frac{\zeta}{4}\right)^M + O\left(\left(\frac{\zeta}{4}\right)^{2M}\right)\right) \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right)^2\right)\right) & \\ \arccos\left(1 - \left(128\left(\frac{\zeta}{4}\right)^{2M} + O\left(\left(\frac{\zeta}{4}\right)^{3M}\right)\right) \left(1 + O\left((|z_1 - 1| + |z_2 - 1|)^2\right)\right)^2\right) & \end{aligned}$$

We finally take the limit $z_i \rightarrow 1$, and get that

$$\begin{aligned} \gamma_{\max} &= \pi - \arccos\left(1 - \left(128\left(\frac{\zeta}{4}\right)^{2M} + O\left(\left(\frac{\zeta}{4}\right)^{3M}\right)\right)\right) = \pi - \sqrt{2} \sqrt{128\left(\frac{\zeta}{4}\right)^{2M} + O\left(\left(\frac{\zeta}{4}\right)^{3M}\right)} \\ &= \pi - 16\left(\frac{\zeta}{4}\right)^M \left(1 + O\left(\left(\frac{\zeta}{4}\right)^{3M}\right)\right). \end{aligned} \tag{26}$$

6. A bound on the quasisymmetric constant

In this section we will give an outline of Świątek's proof of the quasisymmetric property of the conjugacy of the critical circle maps to the rotation from [26]. The goal of our outline will be to recover a useful expression for the constant of quasisymmetry of the conjugacy only in terms of the upper bound on the integers in the continued fraction expansion of the rotation number. We will not give a detailed proof of the Świątek's theorem, our objective will be to extract the necessary dependency.

To obtain a bound on the quasisymmetric constant of the homeomorphism that conjugates a circle homeomorphism with a critical points to the rigid rotation, we will have to obtain several commensurability relations between the intervals bounded by a point and its forward and backward closest returns. This will be done in Lemma 6. The proof of this Lemma uses the following fact (see, for example, [26])

Lemma 5. *Let $g : \mathbb{R} \mapsto \mathbb{R}$ be increasing and 1-periodic. Choose Q positive so that $\rho(g) = P/Q$ in simplest terms, if $\rho(g)$ is rational, or $Q = \infty$, if $\rho(g)$ is irrational. Then for every $x \in \mathbb{R}$, and every pair of integers p and q , $|q| < Q$,*

$$(g^q(x) - x - p)(q\rho(g) - p) > 0.$$

We will now adopt the proof of the commensurability property [26] with the objective of obtaining an explicit expression for the commensurability constant.

Lemma 6. *(Lemma 1.2. from [26]) Suppose that $f : \mathbb{R} \mapsto \mathbb{R}$ is a lifting of a degree 1 circle homeomorphism with an irrational rotation number ρ . Assume that g satisfies a cross-ratio inequality with respect to the cross-ratio module (9) with bound Q . Let p_n/q_n be a convergent of $\rho(f)$. Then there exists K_1 ,*

$$K_1 = 2 \left(1 - \left(\mu^{-1} \left(\frac{1}{8\pi} Q^{-6} \ln \frac{5}{3} \right) \right)^2 \right)^{-6},$$

so that for every $x \in \mathbb{R}$

$$K_1^{-1} |f^{q_n}(x) - p_n - x| \leq |f^{-q_n}(x) + p_n - x| \leq K_1 |f^{q_n}(x) - p_n - x|.$$

Proof. Assume that $p_n/q_n < \rho(f)$. Otherwise we can consider $F(x) = -f(-x) + 1$: the same estimate will follow from a similar argument for F and $-x$ in place of f and x , respectively.

Denote p'/q' the next fraction larger than p_n/q_n from the sequence of the best rational approximants with $q' \leq q_n$. Such p'/q' is larger than $\rho(f)$, otherwise

$$0 < \rho(f) - \frac{p'}{q'} < \rho(f) - \frac{p}{q} \implies |q'\rho(f) - p'| < |q\rho(f) - p|$$

which contradicts the fact that q_n is the closest return time for times smaller or equal to q_n .

Fix a point $x \in \mathbb{R}$. From Lemma 5, $f^{q_n}(x) - p_n > x$ and $f^{q'}(x) - p' < x$, therefore, $f^{-q'}(x) + p' > x$. Choose $n \geq 2$ such that $f^{-q'}(x) + p' \in (f^{(n-1)q_n}(x) - (n-1)p_n, f^{nq_n}(x) - np_n)$.

Next, consider $F_s = f - s$ where s is a non-negative number. As s increases, points in the forward orbit of x shift to the left, those in the backward orbit - to the right. Hence, there is a unique value s^* of s for which

$$F_{s^*}^{nq_n}(x) - np_n = F_{s^*}^{-q'}(x) + p'. \quad (27)$$

For simplicity, we will write F in place of F_{s^*} . It follows from (27), that such F has the rotation number equal to

$$\rho' = \frac{np_n + p'}{nq_n + q'}.$$

We have the following ordering of points (see [26] for details)

$$0 < F^{q_n}(x) - p_n < f^{q_n}(x) - p_n < F^{2q_n}(x) - 2p_n, \quad (28)$$

$$0 > F^{-q_n}(x) + p_n > f^{-q_n}(x) + p_n > F^{-2q_n}(x) + 2p_n. \quad (29)$$

Consider the exponential projection of the real line on the circle: $\pi(x) = e^{2\pi i x}$. The periodic orbit $\pi(F^i(x))$ consists of $nq_n + q'$ points, while the points $\pi(x)$ and $\pi(F^{q_n}(x))$ are consecutive on the circle (again, see [26] for details).

Denote \mathcal{I} the collection of arcs on the circle with endpoints at two consecutive points of the orbit of x by the projection of F . If $I_1, I_2 \in \mathcal{I}$ are adjacent, and F satisfies a cross-ratio inequality with respect to the cross-ratio module (9) with bound Q , then

$$\frac{|I_1|}{|I_2|} \geq \frac{1}{2} \left(1 - \left(\mu^{-1} \left(\frac{1}{8\pi} Q^{-6} \ln \frac{5}{3} \right) \right)^2 \right). \quad (30)$$

This can be seen from the following argument. Choose $I_3 \in \mathcal{I}$ adjacent to I_2 on the opposite side to I_1 . Lift I_1, I_2, I_3 to the line to get some intervals $(a, b), (b, c), (c, d)$ respectively. Observe that $|I_1|/|I_2| > \mathbf{Cr}(a, b, c, d)$. Choose the smallest l such that $f^l(I_2)$ is the shortest arc in \mathcal{I} . The configuration $(a, b, c, d), \dots, (f^l(a), f^l(b), f^l(c), f^l(d))$ has the intersection number at most 3 (see [26]). Also

$$\mathbf{Cr}(f^l(a), f^l(b), f^l(c), f^l(d)) = \frac{|f^l(I_1)| \cdot |f^l(I_3)|}{(|f^l(I_1)| + |f^l(I_2)|) \cdot (|f^l(I_3)| + |f^l(I_2)|)} = \frac{1 \cdot 1}{\left(1 + \frac{|f^l(I_2)|}{|f^l(I_1)|}\right) \cdot \left(1 + \frac{|f^l(I_2)|}{|f^l(I_3)|}\right)} \geq \frac{1}{4}.$$

Therefore, by (10) and by Lemma 3,

$$Q^6 \chi(a, b, c, d) \geq \chi(f^l(a), f^l(b), f^l(c), f^l(d)) \geq \frac{1}{\pi} \ln \frac{5}{3}. \quad (31)$$

Finally, recall the condition (12):

$$\frac{|I_1|}{|I_2|} > \mathbf{Cr}(a, b, c, d) \implies 8\mu \left(\sqrt{1 - \sqrt{\frac{|I_1|}{|I_2|}}} \right) > \chi(a, b, c, d).$$

Putting this together with (31), we obtain

$$\mu \left(\sqrt{1 - \sqrt{\frac{|I_1|}{|I_2|}}} \right) \geq \frac{1}{8\pi} Q^{-6} \ln \frac{5}{3} \implies \frac{|I_1|}{|I_2|} \geq \left(1 - \left(\mu^{-1} \left(\frac{1}{8\pi} Q^{-6} \ln \frac{5}{3} \right) \right)^2 \right)^2 \equiv D,$$

which is just (30). It follows immediately, that the four intervals with the limiting points

$$F^{-2q_n}(x) + 2p_n, F^{-q_n}(x) + p_n, x, F^{q_n}(x) - p_n, F^{2q_n}(x) - 2p_n$$

have lengths comparable with the factor D^{-3} . Together with the inequalities (28) and (29), the Lemma follows with

$$K_1 = 2D^{-3}.$$

□

We will now outline the proof of the quasimetric property of the conjugacy.

Proposition 1. (Prop. 1 in [26]) *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a lifting of degree 1 circle homeomorphism with an irrational rotation number $\rho(f)$. Suppose that f satisfies a cross-ratio inequality with bound Q .*

Then, there exist a lift of degree 1 circle homeomorphism $h : \mathbb{R} \mapsto \mathbb{R}$, which conjugates f to a translation by $\rho(f)$:

$$f(h(x)) = h(x + \rho(f)).$$

Furthermore, if $\rho(f)$ is of bounded type, then h is quasimetric with the quasimetric constant

$$K_2 = \max \{2, K_1^{B+1}\}. \quad (32)$$

Proof. By assumption $\rho(f)$ is of constant type: there exists a positive integer B such that $\rho(f) = [a_1, a_2, \dots]$, $\sup_{i \geq 1} a_i \leq B$. Recall that

$$a_n |q_n \rho(f) - p_n| \leq |q_{n-1} \rho(f) - p_{n-1}| \leq (a_n + 1) |q_n \rho(f) - p_n|$$

for all $n \geq 1$ with the convention $p_0/q_0 = 1/0$. Together with Lemma 5, this leads to

$$|f^{\epsilon(a_n+1)q_n}(x) - \epsilon(a_n+1)p_n - x| > |f^{-\epsilon q_{n-1}}(x) + \epsilon p_{n-1} - x| \quad (33)$$

for all $n \geq 1$, $x \in \mathbb{R}$ and $\epsilon \in \{1, -1\}$,

Take $1 \geq t > 0$, and choose $n \geq 0$ so that

$$|q_{n+1} \rho(f) - p_{n+1}| < t \leq |q_n \rho(f) - p_n|,$$

then

$$h(x+t) \in (f^{-\epsilon q_{n+1}}(h(x)) + \epsilon p_{n+1}, f^{\epsilon q_n}(h(x)) - \epsilon p_n], \quad h(x-t) \in [f^{-\epsilon q_n}(h(x)) + \epsilon p_n, f^{\epsilon q_{n+1}}(h(x)) - \epsilon p_{n+1}] \quad (34)$$

where $\epsilon = (-1)^n$. We can now use relation (33) in the following computation

$$\begin{aligned} \frac{|f^{-\epsilon q_{n+1}}(h(x)) + \epsilon p_{n+1} - h(x)|}{|f^{-\epsilon q_n}(h(x)) + \epsilon p_n - h(x)|} &\leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq \frac{|f^{\epsilon q_n}(h(x)) - \epsilon p_n - h(x)|}{|f^{\epsilon q_{n+1}}(h(x)) - \epsilon p_{n+1} - h(x)|} \\ \frac{|f^{-\epsilon q_{n+1}}(h(x)) + \epsilon p_{n+1} - h(x)|}{|f^{\epsilon(B+1)q_{n+1}}(h(x)) - (B+1)\epsilon p_{n+1} - h(x)|} &\leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq \frac{|f^{-\epsilon(B+1)q_{n+1}}(h(x)) + (B+1)\epsilon p_{n+1} - h(x)|}{|f^{\epsilon q_{n+1}}(h(x)) - \epsilon p_{n+1} - h(x)|} \end{aligned}$$

for all $n \geq 0$. By Lemma 6 any two adjacent intervals with the end points $f^{kq_{n+1}}(h(x)) - kp_{n+1}$ and $f^{(k+1)q_{n+1}}(h(x)) - (k+1)p_{n+1}$ for $k \in \mathbb{Z}$ are comparable with the constant K_1 . Therefore,

$$K_1^{-B-1} \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq K_1^{B+1}$$

for $0 \leq t \leq 1$.

Suppose $t > 1$, and let m be its integer part, then since $h(x+m) = h(x) + m$, we have

$$\frac{m}{m+1} \leq \frac{|h(x+t) - h(x)|}{|h(x) - h(x-t)|} \leq \frac{m+1}{m}.$$

Therefore, h is $\max\{2, K_1^{B+1}\}$ -quasisymmetric. □

We can now finish the proof of the Main Theorem. It follows from the Blaschke model for the Siegel disk, that

$$P_\theta = f \circ m \circ h^{-1} \circ R_\theta \circ h \circ m^{-1} \circ f^{-1}$$

on $\partial\Delta_\theta$. An M -quasiconformal entire mapping is η -quasisymmetric with $\eta(t) = \lambda(M)^{2M} \max\{t^M, t^{1/M}\}$, where the ‘‘circular distortion’’ $\lambda(M)$ is defined as

$$\lambda(M) = \sup_{0 \leq \phi \leq 2\pi} \{|f(e^{i\phi})| : f \in \mathcal{F}\}, \quad \mathcal{F} = \{f : \mathbb{C} \mapsto \mathbb{C}, \quad M\text{-quasiconformal}, \quad f(0) = 0, \quad f(1) = 1\}$$

(cf [5]). It is known that $1 \leq \lambda(M) \leq e^{\pi M}/16$ (cf [5]). We, therefore, have that $g = f \circ m \circ h^{-1}$, m being the Möbius transformation (3), satisfies

$$\frac{|g(x+\delta) - g(x)|}{|g(x) - g(x-\delta)|} \leq \eta \left(\frac{|m(h^{-1}(x+\delta)) - m(h^{-1}(x))|}{|m(h^{-1}(x)) - m(h^{-1}(x-\delta))|} \right) \leq \lambda(M)^{2M} (C(\delta)K_2)^M,$$

where $M = 2K_2 - 1$, K_2 is as in (32) and $C(\delta) \rightarrow 1$ as $\delta \rightarrow 0$.

Now, let $\theta = [a_1, a_2, \dots]$ be a quadratic irrational, such that for all i larger than some $N \geq 1$, we have $a_i = a_{i+s}$. Suppose that α , as in (1), is larger than $1/2$ and the period is even. Then, the bound (15) follows

immediately from 3) of Lemma 4. If α is larger than $1/\sqrt{2}$, but the period is odd, then one can consider the intervals

$$[1, P_{\theta}^{q_{n+2s}}(1)] \subset [1, P_{\theta}^{q_n}(1)],$$

(recall, that $|R_{\theta}^{q_{n+2s}}(1) - 1| = \alpha^2 |R_{\theta}^{q_n}(1) - 1|$), to obtain from 3) of Lemma 4 that

$$\frac{|P_{\theta}^{q_{n+2s}}(1) - 1|}{|P_{\theta}^{q_n}(1) - 1|} \leq \frac{C(n)}{K \left(\frac{1}{1+K^{-1}} \right)^{\log_2 \frac{\alpha^2}{1-\alpha^2}} + 1},$$

K being as in the Main Theorem, and since, as $n \rightarrow \infty$,

$$\frac{\frac{|P_{\theta}^{q_{n+2s}}(1) - 1|}{|P_{\theta}^{q_n}(1) - 1|}}{\left(\frac{|P_{\theta}^{q_{n+2s}}(1) - 1|}{|P_{\theta}^{q_n}(1) - 1|} \right)^2} \rightarrow 1, \quad (35)$$

the bound (13) follows. If the period s is even, and $\alpha \leq 1/2$, then the bound (16) follows from 1) of Lemma 4. The bound (14) follows the same result 1) of Lemma 4, and (35) above.

7. Cross-ratio inequality

In this Section we will recall Świątek's proof of the fact that the cross-ratio module (9) satisfies the cross-ratio inequality with a definite bound. Our goal will be to adopt the proof to the particular case of the Blaschke product, and to show that in this case the bound can be chosen to be "absolute" - independent of the rotation number, as long as it is a quadratic irrational.

Let $g(z) = \frac{1}{2\pi i} \ln(\tilde{Q}_{\theta}(\exp(2\pi iz)))$ be the exponential lift of the modified Blaschke product.

Lemma 7. *The exponential lift of the modified Blaschke product \tilde{Q}_{θ} satisfies the following cross-ratio inequality with respect to the cross-ratio module (9):*

$$\prod_{i=1}^n \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq 8.$$

Proof. The lift g is holomorphic in the upper half-plane, and conformal in all quadrilaterals that do not contain the critical points $\dots, -3, -2, -1, 0, 1, 2, 3, \dots$.

Suppose that (a_i, b_i, c_i, d_i) is some allowable configuration of quadruples of points (in particular, at most one interval (a_i, b_i, c_i, d_i) contains a critical point). We first consider the contribution to the cross-ratio inequality of all non-critical intervals. The map g is conformal on the corresponding quadrilaterals, and we get immediately

$$\prod_{\text{non-critical intervals}} \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} = 1.$$

Now, consider points $(a_{i^*}, b_{i^*}, c_{i^*}, d_{i^*})$ in the critical interval, choose $k > |a_{i^*} - d_{i^*}|$ and a quadrilateral $Q' \subset U_k \cap Q(g(a_{i^*}), g(b_{i^*}), g(c_{i^*}), g(d_{i^*}))$. Its ring domain is contained in U_k , and

$$\text{mod } R(Q') \geq \frac{1 - \frac{|g(a_{i^*}) - g(d_{i^*})|}{k}}{2 \text{ mod } Q(g(a_{i^*}), g(b_{i^*}), g(c_{i^*}), g(d_{i^*}))} \geq \frac{1}{4} \chi(g(a_{i^*}), g(b_{i^*}), g(c_{i^*}), g(d_{i^*})). \quad (36)$$

Consider $R' = g^{-1}(R(Q'))$. If the critical point c lies in (b_{i^*}, c_{i^*}) , then

$$\text{mod } R' = \frac{1}{d} \text{ mod } R(Q'),$$

where d is the degree of branching of g at the critical point. For the modified Blaschke product $d = 2$. If the critical point c lies in (a_{i^*}, b_{i^*}) or (c_{i^*}, d_{i^*}) , then one can find a curve $\gamma \ni g(c)$ that divides the ring domain $R(Q')$ in two ring domains R_{outer} and R_{inner} . If $\text{mod } R_{\text{outer}} > \text{mod } R_{\text{inner}}$, set $R' = g^{-1}(R_{\text{outer}})$, then

$$\text{mod } R' \geq \frac{1}{2d} \text{ mod } R(Q').$$

If $\text{mod } R_{inner} > \text{mod } R_{outer}$, set $R' = g^{-1}(R_{inner})$, then

$$\text{mod } R' \geq \frac{1}{2} \text{mod } R(Q').$$

Therefore, in either case,

$$2d \text{mod } R' \geq \text{mod } R(Q'). \quad (37)$$

Finally,

$$\chi(a_{i^*}, b_{i^*}, c_{i^*}, d_{i^*}) = 2 \text{mod } R(Q(a_{i^*}, b_{i^*}, c_{i^*}, d_{i^*})) \geq 2 \text{mod } R',$$

which, we use, together with (36), in the estimate for the critical interval:

$$\frac{\chi(g(a_{i^*}), g(b_{i^*}), g(c_{i^*}), g(d_{i^*}))}{\chi(a_{i^*}, b_{i^*}, c_{i^*}, d_{i^*})} \leq \frac{4 \text{mod } R(Q')}{2 \text{mod } R'} \leq 4d. \quad (38)$$

We combine (36) with (38) to obtain the bound in the claim of the Lemma.

□

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